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On the Time-Independence of Entropy dimensions associated with a W^*-Dynamical System

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Throughout this paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of all positive integers, the set of all real numbers and the set of all complex numbers, respectively, $\mathcal{H}$ and $\mathcal{B}(\mathcal{H})$ denote a separable Hilbert space and the algebra of all bounded operators on $\mathcal{H}$, respectively. Let $\mathcal{N}_{*,+,1}(\mathcal{B}(\mathcal{H}))$ be the set of all normal states on $\mathcal{B}(\mathcal{H})$. If $\mathcal{S}$ is a weak* compact and convex subset of $\mathcal{N}_{*,+,1}(\mathcal{B}(\mathcal{H}))$, then the set of all extremal points belonging to $\mathcal{S}$, which is denoted by $ex\mathcal{S}$, is non-empty. For any normal state $\phi\in\mathcal{S}$, if there exist both a non-negative sequence $\{\lambda_k; k\in\mathbb{N}\}$ satisfying $\sum_k\lambda_k = 1$ and a sequence of normal states $\{\phi_k; k\in\mathbb{N}\}\subset ex\mathcal{S}$, which enable ϕ to be represented by the following countable convex combination:

$$\phi = \sum_{k=1}^{\infty}\lambda_k\phi_k,$$

then, we define $D(\phi,\mathcal{S})$ by the set of all non-negative sequences that enable ϕ to be represented by the above way. Now, for any positive number $\alpha\neq 1$, Ohya's $(\mathcal{S},\alpha)$-entropy of ϕ is defined by

$$S(\phi,\mathcal{S},\alpha) = inf\left\{\frac{log\sum_{k=1}^{\infty}\lambda_k^{\alpha}}{1-\alpha}; \{\lambda_k; k\in N\}\in D(\phi,\mathcal{S})\right\}.$$

Here, Ohya's $\mathcal{S}$-entropy dimension of ϕ is defined by

$$d(\phi,\mathcal{S}) = inf\{\alpha>0; S(\phi,\mathcal{S},\alpha)<\infty\}.$$

Throughout this paper, we will treat the case that $\mathcal{S} = \mathcal{N}_{*,+,1}(\mathcal{B}(\mathcal{H}))$ holds and we will abbreviate $d(\phi,\mathcal{N}_{*,+,1}(\mathcal{B}(\mathcal{H})))$ to $d(\phi)$ for simplicity.

Let $(\mathcal{B}(\mathcal{H}),\mathbb{R},\alpha)$ be a W^*-dynamical system, and α be a surjective continuous action defined on $\mathbb{R}$ with values in the set of all surjective *-automorphism group on $\mathcal{B}(\mathcal{H})$, that is, for any $s,t\in\mathbb{R}$, $\alpha_s o\alpha_t = \alpha_{s+t}$ holds and α_t is a surjective *-homomorphism defined on $\mathcal{B}(\mathcal{H})$ with values in $\mathcal{B}(\mathcal{H})$ which is continuous in the σ-weak operator topology and satisfies the following condition:

$$lim_{t\to s} < x,\alpha_t(A)y> = < x,\alpha_s(A)y>, \qquad x,y\in\mathcal{H}, \quad A\in\mathcal{B}(\mathcal{H}).$$

Then, it follows from the following theorem that the entropy dimensions of the normal states constructed by the combination with the initial states and the continuous action assocated with the given W^*-dynamical system are time-independent.

Theorem. Let α be a surjective continuous action. Then, for any normal state ϕ, $d(\phi) = d(\phi \circ \alpha_t)$ holds for any $t \in \mathbb{R}$.

Proof. For any $x \in \mathcal{H}$, the vector state constructed by x, which is denoted by ω_x is defined by

$$\omega_x(A) = < x|A|x >, \quad A \in \mathcal{B}(\mathcal{H}).$$

Here, we can assume that ϕ is represented by

$$\begin{aligned} \rho &= \sum_{k=1}^{\infty} \lambda_k |f_k >< f_k|, \\ \phi(A) &= tr(\rho A), \qquad A \in \mathcal{B}(\mathcal{H}), \end{aligned}$$

where $\{\lambda_k\}$ is a non-negative sequence satisfying $\sum_k \lambda_k = 1$, and $\{f_k\}$ is an orthonormal system of $\mathcal{H}$. Then, ϕ can be represented by

$$\phi = \sum_{k=1}^{\infty} \lambda_k \omega_{e_k}.$$

Since $\phi \circ \alpha_t = 0$ implies that $\phi = 0$ holds , $j \neq k$ implies that $\omega_{e_j} \circ \alpha_t \neq \omega_{e_k} \circ \alpha_t$ holds. Therefore, it is sufficient to prove that, for any positive integer k, $\omega_{e_k} \circ \alpha_t$ belongs to $ex\mathcal{N}_{*,+,1}(\mathcal{B}(\mathcal{H}))$ holds. Let ω be an element of $ex\mathcal{N}_{*,+,1}(\mathcal{B}(\mathcal{H}))$ and ψ be $\omega \circ \alpha_t$ and $\{\mathcal{H}_\omega, \pi_\omega, x_\omega\}$ (resp. $\{\mathcal{H}_\psi, \pi_\psi, x_\psi\}$) be the cyclic representation of $\mathcal{B}(\mathcal{H})$ (resp. $\mathcal{B}(\mathcal{H})$) constructed by ω (resp. ψ). Let $(\alpha_t)_{\omega,\psi}$ be an operator on $\{\pi_\psi(B)x_\psi; B \in \mathcal{B}(\mathcal{H})\}$ with values in $\{\pi_\omega(A)x_\omega; A \in \mathcal{B}(\mathcal{H})\}$ defined by

$$(\alpha_t)_{\omega,\psi} \pi_\psi(B) x_\psi = \pi_\omega((\alpha_t)(B)) x_\omega, \quad B \in \mathcal{B}(\mathcal{H}).$$

Then, for any $B, C \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned} < (\alpha_t)_{\omega,\psi} \pi_\psi(B) x_\psi | (\alpha_t)_{\omega,\psi} \pi_\psi(C) x_\psi > &= < \pi_\omega((\alpha_t)(B)) x_\omega | \pi_\omega((\alpha_t)(C)) x_\omega > \\ &= < x_\omega | \pi_\omega((\alpha_t)(B)^*(\alpha_t)(C)) x_\omega > \\ &= < x_\omega | \pi_\omega((\alpha_t)(B^*C)) x_\omega > \\ &= \omega((\alpha_t)(B^*C)) = \psi(B^*C) \\ &= < x_\psi | \pi_\psi(B^*C) x_\psi > \\ &= < \pi_\psi(B) x_\psi | \pi_\psi(C) x_\psi > . \end{aligned}$$

These equalities imply that $(\alpha_t)^*_{\omega,\psi}(\alpha_t)_{\omega,\psi}$ is the identity mapping. It is clear that the uniform closure of $\{\pi_\omega((\alpha_t)(B))x_\omega; B \in \mathcal{B}(\mathcal{H})\}$ is exactly equal to $\mathcal{H}_\omega$, because α_t is surjective. Therefore, $(\alpha_t)_{\omega,\psi}$ can be uniquely extended to an isometry defined on $\mathcal{H}_\psi$. Since, for any $B, C \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned} (\alpha_t)_{\omega,\psi} \pi_\psi(B) (\alpha_t)^*_{\omega,\psi} \pi_\omega((\alpha_t)(C)) x_\omega &= (\alpha_t)_{\omega,\psi} \pi_\psi(B) (\alpha_t)^*_{\omega,\psi} (\alpha_t)_{\omega,\psi} \pi_\psi(C) x_\psi \\ &= (\alpha_t)_{\omega,\psi} \pi_\psi(BC) x_\psi \\ &= \pi_\omega((\alpha_t)(BC)) x_\omega \\ &= \pi_\omega((\alpha_t)(B)) \pi_\omega((\alpha_t)(C)) x_\omega, \end{aligned}$$

these equalities imply that $(\alpha_t)_{\omega,\psi}\pi_\psi(B)(\alpha_t)^*_{\omega,\psi} = \pi_\omega((\alpha_t)(B))$ holds for any $B \in \mathcal{B}(\mathcal{H})$, and

$$\{\pi_\psi((\alpha_t)(B))x_\psi; B\in\mathcal{B}(\mathcal{H})\}' = (\alpha_t)^*_{\omega,\psi}\{\pi_\omega((\alpha_t)(B))x_\omega; B\in\mathcal{B}(\mathcal{H})\}'(\alpha_t)_{\omega,\psi} = \mathbb{C}I,$$

where I means the identity mapping on $\mathcal{H}_\psi$, and $\mathcal{A}'$ means the commutant of an algebra $\mathcal{A}$. These equalities imply that the cyclic representation $\{\mathcal{H}_\psi, \pi_\psi, x_\psi\}$ is irreducible, therefore, we obtain the conclusion.

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